MATHEMATICS SOLUTION
(CBCGS SEM – 4 MAY 2019)
BRANCH – EXTC ENGINEERING

1 a)If X_1 has mean 4 and variance 9& X_2 has mean -2 and variance 4 where X_1&X_2 are (05)independent, find E ($2X_1+X_2-3$) and V($2X_1+X_2-3$). **(5)**

Ans: Given

	X_1	X_2
Mean	**$E(X_1)=4$**	**$E(X_2)=-2$**
Variance	**$V(X_1)=9$**	**$V(X_2)=4$**

Since X_1 ,X_2 are independent, we use following Formulae:

E(a X + b Y +c) = a E(X) + b E (Y) + c and V (a X+bY +c)= a²V(X)+b²V(Y)

∴ E ($2X_1+ X_2-3$) = 2 E (X_1) +E(X_2)-3
= 2(4)+(-2)-3
=3

And,
V($2X_1+X_2-3$) = 2^2 V (X_1) +1^2V (X_2)
=4 (9) +1(4)
=40
Hence, ***E ($2X_1 + X_2 -3$) ;V ($2X_1 + X_2 -3$) =40***

1 b)Find the extremal of $\int_{x_0}^{x_1}(x + y')\, y'\, dx$. **(05)**

Ans. Let $F = (x + y')\, y' = xy' + y'^2$

Since F does not contain y explicitly, by corollary of Euler's Lagrange equation $\frac{\partial F}{\partial y'} = c$

$$\therefore \quad x.1 + 2y'$$

$$\therefore 2\,\frac{dy}{dx} = c - x$$

$$dy = \frac{1}{2}(c - x)dx$$

On integration, $y = \frac{1}{2}\left(cx - \frac{x^2}{2}\right) + c_2$

Put $c_1 = \frac{1}{2}c$,

$\boldsymbol{y = \frac{-1}{4}x^2 + c^1 x + c^2}$,**is the required external.**

1 c) State Cauchy-Schwartz inequality in R^3 and verify it for u= (-4,2,1)&v=(8,-4,-2). (05)

Ans:<u>**Part - I**</u>

<u>Cauchy Schwartz Inequality:</u>

Statement: “If ‘u’ and ‘v’ are vectors in a real inner product space then

$$|\vec{u}.\ \vec{v}| \leq |\vec{u}|\,.|\vec{v}|$$

Proof :

If $\vec{u}$ and $\vec{v}$ are any two vectors in R^2,then by definition of Dot product,

$\vec{u}.\ \vec{v}|\vec{u}|\,.|\vec{v}|\cos\theta$

$\therefore\ |\vec{u}.\ \vec{v}| = |\vec{u}\vec{v}\ \cos\theta| = |\vec{u}|\,.|\vec{v}|\,.|\cos\theta|$

But, $-1 \leq \cos\theta \leq 1$

i.e. $|\cos\theta| \leq 1$

$$\therefore\ |\vec{u}.\ \vec{v}| \leq |\vec{u}|\,.|\vec{v}|$$

Hence, proved.

<u>**Part II:**</u>

Let $u = (-4, 2, 1), v = (8, -4, -2)$.

$= \|u\| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$

$= \sqrt{(-4)^2 + 2^2 + 1^2}$

$=\sqrt{21}$ and

$||v|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

$$= \sqrt{8^2 + (-4)^2 + (-2)^2}$$

$$= \sqrt{84}$$

$\therefore ||\vec{u}||\,||\vec{v}|| = \sqrt{21} \times \sqrt{84}$

$\therefore ||\vec{u}||\,||\vec{v}|| = 42 \rightarrow (1)$

Also ,$\langle u,v \rangle$ = (-4)(8) +(2)(-4)+(1)(-2)

$= -32 - 8 - 2$

=-42

$\therefore |\langle \vec{u}, \vec{v} \rangle|$,=42→ (2)

From (1) & (2),we observe $|\langle \vec{u},\vec{v} \rangle| = ||\vec{u}||\,||\vec{v}||$

∴Cauchy -Schwartz inequality is verified.

2 a)Show the matrix A=$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ is non-derogatory . (06)

Ans: Let λ be Eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$

On Solving we get,

λ^3- (sum of diagonal elements)λ^2 + (sum of minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (2+1-1)\lambda^2 + (-4-5+4)\lambda - (-6) = 0$$

$$\therefore \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

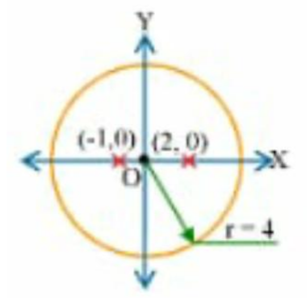
∴Eigen values (λ) are 1,2,-3

Since Eigen Values are distinct, Matrix A is Non-Derogatory.

2 b) Using Cauchy's Residue theorem evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}dz$, where C is |z|=4. (06)

Ans: The circle |z|=4 has centre (0,0) and radius 4.

Here, Z_0 =-1 is a pole of order 2 and z_0 = 2 is a simple pole.



Z_0 = - 1 and Z_0 =2 Lies inside the circle.

R_1= Residue of f(z) at " Z_0 = -1"

$=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n \text{ x } f(z)$

$= \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}(z+1)^2 \text{x} \frac{z-1}{(z-1)^2(z-2)}$

$= \frac{1}{1!}\lim_{z\to -1}\frac{(z-2).1(z-1).1}{(z-2)^2}$

$=\frac{-1}{(-1-2)^2}$

$=\frac{-1}{9}$

R_2 = Residue of f(z) at "z_0 = 2"

$= \frac{lim}{z\to z_0}(z-z_0)\text{x } f(z)$

$=\frac{lim}{z\to 2}(z-2) \text{ x } \frac{z-1}{(z+1)^2(z-2)}$

$= \frac{2-1}{(2+1)^2}$

$=\frac{1}{9}$

By Cauchy's Residue theorem,$\int f(z)dz=2\pi i(R_1+R_2+\ldots)$

$\therefore \int_c \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i \cdot \left(\frac{-1}{9} + \frac{1}{9}\right)$

$= \int_c \frac{z^2}{(z-1)^2(z+1)}dz = 0$

2 c) Show that the extremal of isoperimetric problem $I = \int_{x_1}^{x^2} (y')^2\, dx$ subject to the condition $\int_{x^1}^{x^2} y\, dx = k$ is a parabola. **(08)**

Ans:Let $\int_{x^1}^{x^2} F dx = \int_{x^1}^{x^2} (y')^2$ dx and $\int_{x^1}^{x^2} G dx = \int_{x^1}^{x^2} y dx$ =k

∴ F = y'^2 and G=y

∴ Langrangian function H =F + λ G

∴H = $y'^2 + \lambda y$→ (1)

∴ $\frac{\partial H}{\partial y'}$=2y' →(2)

And,$\int_{x^1}^{x^2} H\, dx = \int_{x^1}^{x^2} (y'^2 + \lambda y) dx$

Since the above integral does not contain 'x' explicitly,by corollary of Euler's Language equation,

$H - y'\frac{\partial H}{\partial y'} = c$

∴ $y'^2 + \lambda y - y'.2y' = c$ (From 1&2)

∴ $-y'^2 + \lambda y = c$

∴ $y'^2 = \lambda y - c$

∴ $y' = \frac{dy}{dx} = \sqrt{\lambda y - c}$

$$\therefore \frac{1}{\sqrt{\lambda y} - c}\, dy = dx$$

$$\therefore \frac{1}{\sqrt{\lambda\sqrt{y}-c/\lambda}}dy = dx$$

$\therefore \frac{1}{\sqrt{\lambda}}\left(y-\frac{c}{\lambda}\right)^{-1/2}$dy=dx

On integration, $\frac{1}{\sqrt{\lambda}} \times \frac{(y-c/\lambda)^{1/2}}{1/2}$ = x+c_1

$\therefore \left(y-\frac{c}{\lambda}\right)^{1/2} = \frac{\sqrt{\lambda}}{2}(x+c_1)$

$\therefore$**On squaring,** $y-\frac{c}{\lambda} = \frac{\lambda}{4}(x+c_1)^2$

$\therefore$**y=$\frac{\lambda}{4}(x+c_1)^2 + \frac{c}{\lambda}$, which is a parabola.**

2 d) Is the matrix A=$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable? If so find the diagonal form and the transforming matrix. (08)

Ans.Let λ be eigen value of matrix A.
Characteristic equation is $|A-\lambda I|=0$

$\therefore \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix}=0$

On Solving ,we get

λ^3-(sum of diagonal elements) λ^2+ (sum 0f the minors of diagonal elements) λ- |A|=0

$\therefore \lambda^3-(2+2+1)\lambda^2+(2+2+3)\lambda-3=0$
$\therefore \lambda^3-5\lambda^2+7\lambda-3=0$
$\therefore$ Eigen values (λ) are $1, 1, 3$

Case 1: $\lambda=1$
$\therefore [A-\lambda I]X=0$

$$\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\0 & 0 & 0\end{bmatrix}.\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

R_2-R_1⇒ $\begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow (1)$

Number of unknowns(n)=3
Rank (r) = number of non-zero rows=1

Algebraic multiplicity (A.M)
=No.of times "λ=1" is repeated =2

Geometric multiplicity (G.M)$= n - r = 3 - 1 = 2$
∴ A.M=G.M for "λ =1"
Expanding (1),$x_1+x_2+x_3=0$
Put x_2= t and x_3=s
∴x_1+ t +s =0
∴$x_1 = -t - s$
∴ Eigen vector X =$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-t-s\\t\\s\end{bmatrix}= -\begin{bmatrix}1\\-1\\-1\end{bmatrix}t - \begin{bmatrix}1\\0\\-1\end{bmatrix}$s

∴ Eigen vector X_1=$[1 \; -1 \; 0]$' &X_2=[1 0 -1]'

Case 2: $\lambda = 3$
∴ $[A - \lambda\, I]$X=0
$$\begin{bmatrix}-1 & 1 & 1\\1 & -1 & 1\\0 & 0 & -2\end{bmatrix}.\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

R_2+R_1⇒ $\begin{bmatrix}-1 & 1 & 1\\0 & 0 & 2\\0 & 0 & -2\end{bmatrix}.\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

R_3+R_2 ;0.5R_2 ⇒ $\begin{bmatrix}-1 & 1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}.\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow (3)$
Rank (r)=2

A.M=No.of times "λ=3" is repeated=1

$$G.M = n - r = 3 - 2 = 1$$

∴ A.M=G.M for "λ =3"

Expanding(3),x_3=0 and,

$-x_1+x_2+x_3=0$

∴ $-x_1+x_2+0=0$

∴$x_1=x_2$

∴ $Eigen\ vector\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$=$\begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix}$=$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$x²

∴ Eigen vector X_3 = [1 1 0]'

Since, A.M=G.M for all eigen values, matrix A is diagonalizable.

∴ $M^{-1}AM=D$

∴ Matrix A is diagonalized to Diagonal Matrix D by the Transforming Matrix M, where

D=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$and M=$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

3 a) Verify the Cayley-Hamilton Theorem for matrix A and hence find (06)

$A^{-1}$and A^4 where A =$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$.

Ans: <u>Part I:</u>

Let λ be eigen value of matrix A.

Characteristic equation is |A- λI |=0

∴ $\begin{vmatrix} \lambda - 1 & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix}$=0

On Solving we get,

λ – (sum of diagonal elements)λ^2+ (sum of the minors of diagonal elements) λ- |A|=0

∴ $\lambda^3 - (1 - 1 - 1)\ \lambda^2 + (-3 - 10 - 5)\ \lambda - 40$=0

∴ λ^3+ λ^2- 18 λ – 40=0

Cayley Hamilton Theorem states that the characteristic equation is satisfied by matrix A.

$\therefore A^3+A^2-18A-40I=0 \rightarrow (1)$

Now, $A^2 = A \times A$

$$=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}\times\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}=\begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}$$

$A^3=A^2\times A$

$$=\begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}\times\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}=\begin{bmatrix}44 & 33 & 46\\ 24 & 13 & 74\\ 52 & 14 & 8\end{bmatrix}$$

$\therefore$ LHS $= A^3 + A^2 - 18A - 40I$

$$=\begin{bmatrix}44 & 33 & 46\\ 24 & 13 & 74\\ 52 & 14 & 8\end{bmatrix}+\begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}-18\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}-40\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0=\text{RHS}$$

$\therefore$ **Cayley Hamilton Theorem is verified.**

Part II:

Pre-multiply(1)by A^{-1},

$\therefore A^{-1}(A^3)+A^{-1}(A^2)-18A^{-1}A-40A^{-1}I=0$

$\therefore A^2+A-18I-40A^{-1}=0 \{\because A^{-1}A=I; A^{-1}I=A^{-1}\}$

$\therefore 40A^{-1}=A^2+A-18I$

$$=\begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}-18\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\therefore 40A^{-1}=\begin{bmatrix} -3 & 5 & 11 \\ 14 & -14 & 2 \\ 5 & 5 & -5 \end{bmatrix}$

$\therefore A^{-1}=\frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$

Part III:

Pre-multiply (1)by A,

$\therefore A(A^3)+A(A^2)-18\,A\,(A)-40A(I)=0$

$\therefore A^4+A^3-18A^2-40A=0$

$\therefore A^4= 18A^2+40A-A^3$

$=18\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}+40\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}-\begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$

$\therefore \mathbf{A^4}=\begin{bmatrix} \mathbf{248} & \mathbf{101} & \mathbf{218} \\ \mathbf{272} & \mathbf{109} & \mathbf{50} \\ \mathbf{104} & \mathbf{98} & \mathbf{204} \end{bmatrix}$

3 b)Check whether the following is a subspace of R^3
W={(a,0,0)|a ∈ R} **(03)**

Ans:Let $\underset{u}{\rightarrow}$ and $\underset{v}{\rightarrow}$ be any two vectors in W and let 'k' be any real scaler.

If W is non- empty subset of V then W is sub-space if

a) $\vec{u}+\vec{v}$ is in W.

b) $k\vec{u}$ is in W.

W={(a,0,0)|a ∈ R}
Let $\vec{u}$ = (a_1,0,0)and $\vec{v}$ = (a_2,0,0)be any two vectors belonging to set W such that $a_1, a_2 \in R$

(a) $\vec{u}+\vec{v}$= (a_1,0,0)+(a_2,0,0)

$\therefore \vec{u}+\vec{v} = (a_1 + a_2, 0, 0)$

Sum of two real number is a real number . $\therefore$ $a_1 + a_2 \in R$

$\therefore$ $\vec{u}+\vec{v}$ is in W.

b)k$\vec{u}$ = k(a_1, 0,0)

$\therefore$k$\vec{u}$ = (k a_1,0,0)

Product of two real number $\therefore$k $a_1 \in$ R

$\therefore$k$\vec{u}$ is in W

Hence, set W is a Subspace of R^3

3 c) Check whether the following is a subspaces of R^3:W={(x,y,z)|x=1,z=1,y$\in$ R}. (03)

Ans:Let $\vec{u}$ and $\vec{v}$ be any two vectors is W and let 'k' be any real scalar.

If W is non-empty subset of V then W is sub-space if

a) $\vec{u}+\vec{v}$ is in W.
b) k$\vec{u}$ is in W.

W={(x,y,z)|x = 1,z=1,y$\in$ R}.

Let $\vec{u}$= (x_1,y_1,z_1)=(1,y_1,1)and $\vec{v}$= (x_2,y_2,z_2)= (1,y_2,1) be any two vectors belonging to set W such that y_1, $y_2 \in$ R

a) $\vec{u}+\vec{v}$=(1,y_1,1)+ (1,y_2,1)

$\therefore \vec{u}+\vec{v}$ = (2,y_1+y_2,2)

However,$\vec{u}+\vec{v}$ is not in the form of (1,y,1).

$\therefore \vec{u}+\vec{v} \notin$ W

b) k $\vec{u}$= k(1,y_1,1)

$\therefore$ k $\vec{u}$ = (k,ky_1,k)

However,k $\vec{u}$ is not in the form of (1,y,1).

$\therefore k\vec{u} \notin W$

Hence ,set W is not a subspace of R^3.

3 d) Expand $f(z)=\frac{1}{(z-1)(z-2)}$ in Taylors & Laurent's series indicating regions of convergence. **(08)**

Ans: $f(z) = \frac{1}{(z-1)(z-2)}$

$= \frac{1}{z-2} - \frac{1}{z-1}$ → (1)(By partial Fractions)

Taylor Series:

Let a =0

From (1), $f(z)=(z-2)^{-1}-(z-1)^{-1}$

$\therefore f(a)=f(0)=(0-2)^{-1}-(0-1)^{-1}=\frac{1}{2}$

$\therefore f'(z) = -(z-2)^{-2}+(z-1)^{-2}$

$\therefore f'(a)=f'(0) = -(0-2)^{-2}+(0-1)^{-2} = \frac{3}{4}$

$\therefore f''(z) = 2(z-2)^{-3} -2(z-1)^{-3}$

$\therefore f''(a)=f(0) =2(0-2)^{-3}-2(0-1)^{-3}=\frac{7}{4}$

$\therefore f'''(z)=-6(z-2)^{-4} + 6(z-1)^{-4}$

$\therefore f'''(a)= f'''(0)= -6(0-2)^{-4} + 6(0-1)^{-4} = \frac{45}{8}$

By Taylor Series,

$$f(z) = f(a)+ (z-a) f'(a)+(z-a)^2\frac{f''(a)}{2!} + ...$$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{2} + (z-0).\frac{3/4}{1!} +(z-0)^2.\frac{7/4}{2!} +(z-0)^3.\frac{45/8}{3!} +.....$$

$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \ldots..$

Laurent's series expansion:

We consider following three cases.

Case 1 For |z|< 1,

Obviously, $|z| < 2$

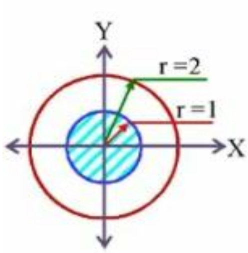
$\therefore |z| < 1$ and $\left|\frac{z}{2}\right| < 1$

$\therefore$ From (1), $f(z) = \frac{1}{2(z/2-1)} - \frac{1}{(z-1)}$

$\therefore \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$

$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right)+(1+z+z^2+z^3+\ldots)$

$= -\left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots\right)+(1+z+z^2+z^3+\ldots)$



Region of Convergence:

Above series is convergent for

$|z| < 1$ and $\left|\frac{z}{2}\right| < 1$ i.e. $|z| < 2$

i.e. $|z|<1$,which is the interior of the circle with centre (0,0) and radius 1.

Case 2: For $1< |z|<2$

$\therefore 1 < |z|$ and $|z| < 2$

$\therefore \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

∴ From (1), $f(z) = \frac{1}{2(z/2-1)} - \frac{1}{z(1-1/z)}$

$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$

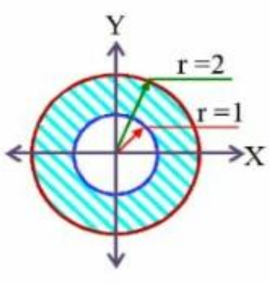
$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right) - \frac{1}{2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$

$= -\left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots\right) + \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots\right)$

Region of Convergence:

Above series is convergent for $\left|\frac{1}{z}\right| < 1$ & $\left|\frac{z}{2}\right| < 1$

i.e.$1 < |z|$ and $|z| < 2$



i.e.$1 < |z| < 2$,which is the annular region between the concentric circles with centre (0,0) and radii 1 & 2

<u>Case 3:</u> For $|z| > 2$,

Obviously, $|z| > 1$
∴ $1 < |z|$ and $2 < |z|$

∴ $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$

∴ From(1), $f(z) = \frac{1}{z(1-2/z)} - \frac{1}{z(1-1/z)}$

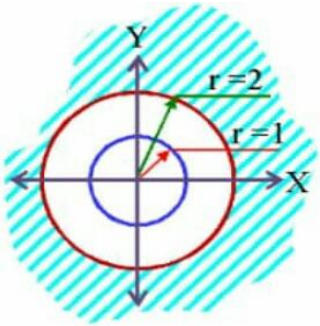
$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$

$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$

$= \left(\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\frac{2^3}{z^4}+\cdots\right)-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots\right)$

Region of Convergence:

Above series is convergent for



$\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$

$\therefore$ 1< |z| and2<|z|

$\therefore$ |z|> 1 and |z| > 2

i.e.|z|>2,which is the exterior region of the circle with centre (0,0) and radius 2.

4 a) Using Rayleigh -Ritz method to solve the boundary value problem **(06)**
$I = \int_0^1 (2xy + y^2 - y^2)dx, 0 \le X \le 1$

Giveny(0)=0,y(1)=0.

Ans :Rayleigh Ritz method:

Let I $\int_0^1 (2xy + y^2 - y'^2)dx \to (1)$

Let the approximate solution be
$Y(x)=c_0+c_1x+c_2x^2 \to$ (2)

Put x=0,y(0)=c_0+ 0 +0 [$\because$ y(0)=0]

$\therefore c_0 = 0 \rightarrow (3)$

Put x = 1 and $c_0 = 0$ in (2), $y(1) = 0 + c_1 + c_2$

$\therefore 0 = c_1 + -c_2$ [Given, y (1) =0]

$\therefore c_2 = -c_1 \rightarrow$ (4)

Substituting (3) and (4) in (2),we get

$Y = 0 + c_1 x - c_1 x^2 \rightarrow$ (5)

Differentiation w.r.t. 'x', $y' = c_1 - 2c_1 x \rightarrow (6)$

Substituting (5) and (6)in (1),we get

$$I = \int_0^1 [2x(c_1x - c_1x^2) + (c_1x - c_1x^2)^2 - (c_1 - 2c_1x)^2]\, dx$$

$$= \int_0^1 [2c_1x^2 - 2c_1x^3 + (c_1^2x^2 - 2c_1^2x^3 + c_1^2x^4) - (c_1^2 - 4c_1^2x + 4c_1^2x^2)]dx$$

$$= \left[\frac{2c_1x^3}{3} - \frac{2c_1x^4}{4} + \frac{c_1^2x^3}{3} - \frac{2c_1^2x^4}{4} + \frac{c_1^2x^5}{5} - c_1^2x + \frac{4c_1^2x^2}{2} - \frac{4c_1^2x^3}{3}\right]_0^1$$

$$= \left[\frac{2c_1}{3} - \frac{c_1}{2} + \frac{c_1^2}{3} - \frac{c_1^2}{2} + \frac{c_1^2}{5} - c_1^2 + 2c_1^2 - \frac{4c_1^2}{3}\right] - [0 - 0 - 0 + 0 - 0 - 0 + 0 - 0]$$

$$\therefore I = \frac{1}{6}c_1 - \frac{3}{10}c_1^2$$

For maximum or minimum, $\frac{dI}{dc_1} = 0$

$$\therefore \frac{dI}{dc_1} = \frac{1}{6} - \frac{3}{10} \times 2c_1 = 0$$

$$\therefore \frac{1}{6} = \frac{3}{5}c_1$$

$$\therefore c_1 = \frac{5}{18}$$

$\therefore$ From (5), $y = \frac{5}{18}x - \frac{5}{18}x^2$

∴ **By Rayleigh Ritz method the approximate solution is y =** $\frac{5}{18}x\,(x-1)$

4 b) If A = $\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ **then prove that 3 tan A = A tan 3.** **(06)**

Ans: Let λ be eigen value of matrix A.

Characteristic equation is |A- λ I|=0

∴ $\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix}$ =0

On Solving ,we get, λ^2- (sum of diagonal elements) λ + |A|=0

∴ λ^2- (-1+1) λ+(-1-8-)=0

∴ λ^2-9=0

∴ $\lambda = \pm 3$

∴ **Eigen values (λ) are 3,-3.**

Since, **matrix is of order 2 x 2**,let tan **A= a A + b I** (where a, b are constants) →(1)

We assume equation (1)is satisfied by eigen value λ

∴tan λ =a λ+ b →(2)

When λ =3,tan 3 = 3 a+b → (3)

When λ = -3,tan(-3) = - 3 a + b

∴ −tan 3 = - 3 a + b → (4)

Adding (3) and (4),0=0 +b

∴ b=0

From (3),tan 3 =3 a + 0

∴ a = $\frac{\tan\quad A}{3}$

Substituting ‘a’ and ‘b’ in (1), tan A = $\frac{\tan \ \ 3}{3}$x A + 0

∴3 tan A= A tan 3

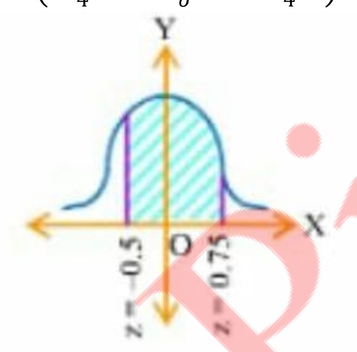
4 c) If sizes of 10000 items are normally distributed with mean 20 cm & standard deviation of 4cm ,find the probability that an item selected at random will have size (08)

- **i) between 18cm and 23cm,**
- **ii) above 26cm**

Ans: Mean (m)=20cm
Standard deviation (σ) =4cm
N = 10000
Let X denote the size of an item.

(i)P (Selected item has size between 18 and 23 cm)= P (18< X < 23)

=P$\left(\frac{18-20}{4} < \frac{X-m}{\sigma} < \frac{23-20}{4}\right)$



=P(-0.5<z < 0.75)

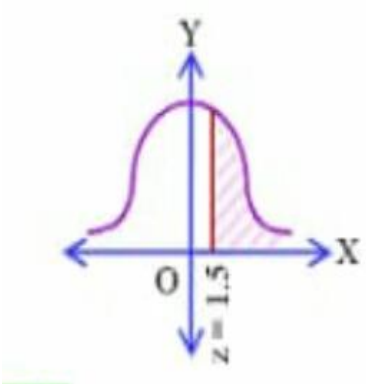
=Area between ‘z = 0’ to ‘z = 0.75’ + Area between ‘z=0’ to ‘z = - 0.5’

=0.2734+0.1915

=0.4649

ii)P(selected item has size above 26 cm) = P (x> 26)

$=P\left(\frac{x-m}{\sigma} > \frac{26-20}{4}\right)$



=P(z> 1.5)
=0.5 - Area between 'z=0'to 'z=1.5'
=0.5 – 0.4332
= 0.0668

Probability that an item selected at random will have size between 18cm and 23 cm = 0.4649
Probability that an item selected at random will have size above 26cm= 0.0668

5 a) Find orthonormal basis of R^3 using Gram-Schmidt process where S = { (1, 0, 0), (3, 7, -2), (0, 4, 1). (6)

Ans.<u>Gram Schmidt orthogonalization:</u>

Let u_1=(1,0,0);u_2 =(3,7,-2); u_3 = (0,4,1);

<u>S1:</u>
Let $v_1 = u_1$ = (1,0,0)
$\therefore \|v_1\|^2 = (1)^2 + (0)^2 + (0)^2 = 1$ and

$\langle u_2, v_1\rangle$ = (3) (1) + (7) (0) + (-2) (0) = 3 + 0 + 0 =3

<u>S2:</u>

Let $v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|} v_1$

$=(3,7,-2) - \frac{3}{1} \times (1,0,0)$

$= (3,7,-2)-(3,0,0)$

$= (0,7,-2)$

$\therefore \|v_2\|^2 = (0)^2 + (7)^2 + (-2)^2 = 53;$

Now,

$\langle u_3, v_1\rangle=(0)(1) + (4)(0) + (1)(0) = 0$

and,

$\langle u_3, v_2\rangle=(0)(0) + (4)(7) + (1)(-2) = 26$

S3:

Let $v_3 = u_3 - \frac{\langle u_3,v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3,v_2\rangle}{\|v_2\|^2} v_2$

$= (0, 4, 1) - \frac{0}{1}(1,0,0) - \frac{26}{53}(0,7,-2)$

$= (0, 4, 1) - \left(0, \frac{182}{53}, \frac{-52}{53}\right)$

$=\left(0, \frac{30}{53}, \frac{105}{53}\right)$

$\therefore \|v_3\|^2 = (0)^2 + \left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2 = \frac{225}{53};$

$\therefore$ **Orthonormal bases are**

$\vec{q_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{1}}(1,0,0) = (1,0,0)$

$\vec{q_2} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{53}}(0,7,-2) = \left(0, \frac{7\sqrt{53}}{53}, \frac{-2\sqrt{53}}{\sqrt{53}}\right)$

And, $\vec{q_3} = \frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{1}{1/\sqrt{225/53}}\left(0, \frac{30}{53}, \frac{105}{53}\right)$

$= \frac{\sqrt{53}}{15}\left(0, \frac{30}{53}, \frac{105}{53}\right) = \left(0, \frac{2\sqrt{53}}{53}, \frac{7\sqrt{53}}{53}\right)$

Orthonormal basis of the subspace S are

$$\left\{(\mathbf{1},\mathbf{0},\mathbf{0}); \left(\mathbf{0}, \frac{\mathbf{7\sqrt{53}}}{\mathbf{53}}, \frac{\mathbf{-2\sqrt{53}}}{\sqrt{\mathbf{53}}}\right), \left(\mathbf{0}, \frac{\mathbf{-2\sqrt{53}}}{\sqrt{\mathbf{53}}}, \frac{\mathbf{7\sqrt{53}}}{\mathbf{53}}\right)\right\}$$

5 b) In a factory, machines A, B & C produce 30%, 50% and 20% of the total production of an item. Out of their production 80%, 50% & 10% are defective respectively. An item is chosen at random and found to be defective .What is the probability that it was produced by machine A. (06)

Ans:

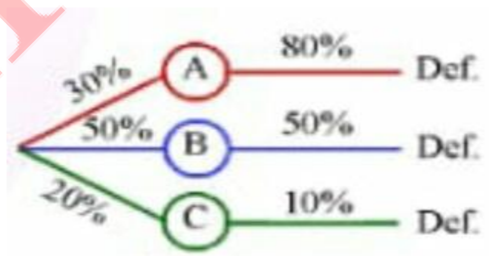
Let A, B and C be the events that an item is produced by machines A, B, C respectively.

P(A)=30% = 0.3; P(B) = 50% = 0.5; P (C) = 20% =0.2;
Let D be the event item a defective item is selected .

Probability an item is defective given that it is manufactured by factory A= P(D/A)=80%=0.8
Probability an item is defective given that it is manufactured by factory B= P(D/B)=50%=0.5
Probability an item is defective given that it is manufactured by factory C= P(D/C)=10%=0.1

Since it is already knows that the item is defective and we want to find the Probability that it is manufactured by factory A, it is problem of reverse probability.
We use Bayes Theorem.

Consider, $P(A)\times P(D/A) + P(B)\times P(D/B) + P(C)\times P(D/C) = 0.3\times 0.8 + 0.5\times 0.5 + 0.2\times 0.1 = 0.51$



$\therefore$ P (A/D) = $\frac{P(A)\times P(D/A)}{P(A)\times P(D/A)+P(B)\times P(D/B)+P(C)\times P(D/C)}$

$$= \frac{0.3\times 0.8}{0.51}$$

$=\frac{24}{51}$ = 0.4706

Hence, the probability that a selected defective item is produced by factory A=0.4706

5 c) Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+9)}dx$ (08)

S1. Consider the contour of a large semicircle with diameter on real axis, centre at origin and above the real axis.

S2.

Let f (z) $=\frac{1}{(z^2+4)(z^2+9)}$

As $z \to \infty, zf(z) \to 0$

S3.

For singularity, $(z^2+4)(z^2+9)=0$

$\therefore z^2 + 4 = 0$ or $z^2 + 9=0$

$\therefore z^2 = -4$ or $z^2 =-9$

$\therefore z^2= 2^2i^2$ or $z^2 = 3^2i^2$

$\therefore z = \pm 2$ i or $z = \pm 3$ i

Here, z_0 = -2 i, 3 i Lies outside while z_0 = 2 i,3i lies inside the contour.

z_0= "2i" and "3i" are simple poles .

S4.

R_1 = Residue of f (z) at "z = 2i"

$= \lim_{z \to z_0} (z-z_0) \times f(z)$

$=\lim_{z \to z_i} (z-2i) \times \frac{1}{(z-2i)(z+2i)(z^2+9)}$

$= \frac{1}{(2i+2i)(2^2i^2+9)}$

$=\frac{1}{4i \times 5}$

$= \frac{1}{20i}$

Similarly,

R_2 = Residue of f (z) at "z=3i"

$= \lim_{z \to 3i} \quad (z-3i) \times \frac{1}{(z-3i)(z+3i)(z^2+4)}$

$= \frac{1}{(3i+3i)(3^2i^2+4)}$

$= \frac{1}{6i \times -5}$

$= \frac{-1}{30i}$

S5.

By Cauchy's Residue theorem ,

$\int_0 \quad f(z)\,dz = 2\pi i\,(R_2 + R_2 + \ldots.)$

$\therefore \int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+9)} dx = 2\pi i \cdot \left\{\frac{1}{20i} - \frac{1}{30i}\right\}$

$\therefore \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = 2\,\pi i \times \frac{1}{10\ \ i}\left\{\frac{1}{2} - \frac{1}{3}\right\}$

$\therefore \int_{-\infty}^{\infty} \frac{\mathbf{1}}{(x^2+4)\,(x^2+9)} dx = \frac{\pi}{30}$

6 a) Evaluate $\int_c \frac{1}{z^3(z+4)}dz$, where C is the circle

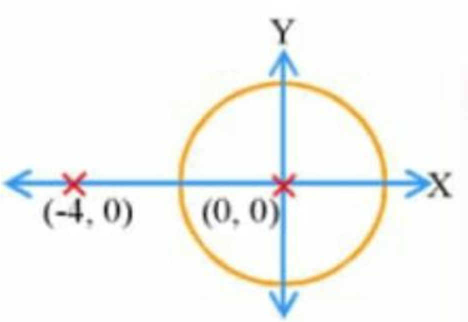
(06)

- **i) |z| = 2 and**
- **ii) |z-3|=2.**

Ans: $\int_c \frac{1}{z^3(z+4)}dz$

Here, $z_0 = 0$ is a pole of order 3 while z_0 = -4 is a simple pole

i) The Circle |z|=2 has centre (0,0) and radius 2



Here, z_0= - 4 lies outside while z_0=0 lies inside the circle.

R_1 = Residue of f (z) at "z=0"

$$=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n \times f(z)$$

$$=\frac{1}{(3-1)!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[z^3 \times \frac{1}{z^3(z+4)}\right]$$

$$=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}(z+4)^{-1}$$

$$=\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}[-1.(z+4)^{-2}]$$

$$=\frac{1}{2}\lim_{z\to 0}[2.(z+4)^{-3}]$$

$$=\frac{1}{2}.2.\frac{1}{(0+4)^3}$$

$$=\frac{1}{64}$$

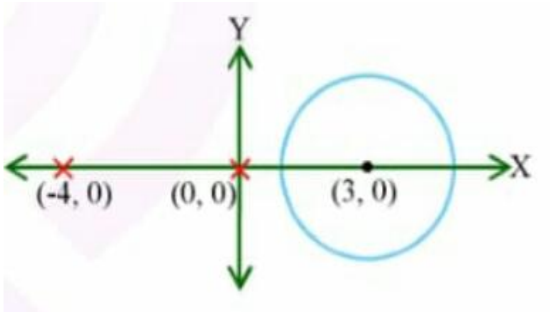
By Cauchy's Residue theorem,

$$\int_c f(z)dz = 2\pi i\,(R_1+R_2+\ldots.)$$

$$\therefore \int_c \frac{1}{z^3(z+4)}dz = 2\pi i.\left(\frac{1}{64}\right)$$

$$\therefore \int_c \frac{1}{z^3(z+4)}dz = \frac{\pi i}{32}$$

ii) Circle |z-3|=2 has centre (3,0) and radius 2



Here, z_0= -4 and z_0= 0 lies outside the circle.

∴ By Cauchy 's Integral theorem , $\int_c f(z)\,dz = 0$

∴ $\int_c \frac{1}{z^3(z+4)} dz = 0$

6 b) Two unbiased dice are thrown three times, using Binominal distribution find the probability that the sum would be obtained. (06)

- **i) once,**
- **ii) twice**

Ans:When a pair of fair dice is thrown,
Sample space

S={(1,1);(1,2);(1,3);(1,4);(1;5);(1;6);
(2,1);(2,2);(2,3);(2,4);(2;5);(2;6);
(3,1);(3,2);(3,3);(3,4);(3;5);(3;6);
(4,1);(4,2);(4,3);(4,4);(4;5);(4;6);
(5,1);(5,2);(5,3);(5,4);(5;5);(5;6);
(6,1);(6,2);(6,3);(6,4);(6;5);(6;6)}

∴n(S)=36

Let A be the event to get the sum of two numbers on dice as 9.

∴ A = {(6,3);(5,4);(4,5);(3,6);}
∴n(A)=4

∴ p-P(A) = $\frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$

∴ q=1 – p=1-$\frac{1}{9} = \frac{8}{9}$

Since, two dice are thrown three times, n=3

Let X denotes number of times the sum nine would be obtained.

For Binominal distribution, P (X=x)= $^nC_x p^x q^{n-x}$

$= {}^3C_x\left(\frac{1}{9}\right)^x \left(\frac{8}{9}\right)^{3-x}$

(i) Probability that the sum nine would be obtained once = P (x = 1)

$= {}^3C_1 \left(\frac{1}{9}\right)^1 \left(\frac{8}{9}\right)^{3-1}$

$= 3 \times \frac{1}{9} \times \frac{64}{81}$

$= \frac{64}{243}$

ii) Probability that the sum nine would be obtained twice = P (X=2)

$= {}^3C_2\left(\frac{1}{9}\right)^2 \left(\frac{8}{9}\right)^{3-2}$

$= 3 \times \frac{1}{81} \times \frac{8}{9}$

$= \frac{8}{243}$

Hence,
i)Probability that the sum nine would be obtained once =$\frac{64}{243}$

ii)Probability that the sum nine would be obtained twice = $\frac{8}{243}$

6 c) A chemical engineer is investigating the effect of process operating temperature X on (08) product yield Y. The results in the following data.

X	**100**	**110**	**120**	**130**	**140**	**150**	**160**	**170**	**180**	**190**
Y	**45**	**51**	**54**	**61**	**66**	**70**	**74**	**78**	**85**	**89**

Find the equation of regression line which will be enable to predict yield on the basis of Temperature. Find also the correlation coefficient between X & Y.

Ans: Let a = 145, b= 67,c=10,c′ = 1

Here n=4

X	Y	u = X -145	v =Y-67	u^2	u v	v^2
100	45	-45	-22	2025	990	484
110	51	-35	-16	1225	560	256
120	54	-25	-13	625	325	169
130	61	-15	-6	225	90	36
140	66	-5	-1	25	5	1
150	70	5	3	25	15	9
160	74	15	7	225	105	49
170	78	25	11	625	275	121
180	85	35	18	1225	630	324
190	89	45	22	2025	990	484
	Total	**0**	**3**	**8250**	**3985**	**1933**

Here n =10

$\vec{x}$ = a + c$\vec{u}$ =a + c .$\frac{\sum u}{n}$= 145 + .$\frac{0}{10}$ = 145

$\vec{y}$ = b + c $\vec{v}$=b + c .$\frac{\sum v}{n}$= 67 +1 x $\frac{3}{10}$ = 67.3

$b_{yx} = b_{vu}$=$\frac{n\sum uv-\sum u\sum v}{n\sum u^2-(\sum u)^2}$ =$\frac{10(3985)-(0)(3)}{10(8250)-(0)^2}$ = 0.4830

$b_{xy} = b_{vu}$=$\frac{n\sum uv-\sum u\sum v}{n\sum v^2-(\sum v)^2}$ =$\frac{10(3985)-(0)(3)}{10(1933)-(3)^2}$ = 2.0616

∴ Regression equation of Y on X is

$Y - \vec{y} = b_{yx} (x - \vec{x})$

∴ y- 67.3= 0.483(x-145)

∴ y = 0.483 x -2.735

Karl Pearson's coefficient of correlation

$r_{xy} = \sqrt{b_{yx} \times b_{xy}}$

$= \sqrt{0.4830 \times 2.0616}$

$= \frac{39850}{\sqrt{82500}\sqrt{19321}}$

= 0.9979

Hence,
Regression Coefficients are
b_{yx} = 0.4830 and b_{xy} = 2.0616

Correlation Coefficient between X & Y is r = 0.9981

Regression equation line to predict yield on the basis of Temperature is
V = 0.483x -2.735